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Holographic noncommutativity

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Abstract: We examine noncommutative Yang-Mills and open string theories using magnetically and electrically deformed supergravity duals. The duals are near horizon regions of Dp-brane bound state solutions which are obtained by using $\mathrm{O}(p+1,p+1)$ transformations of Dp-branes. The action of the T-duality group implies that the noncommutativity parameter is constant along holographic RG-flows. The moduli of the noncommutative theory, i.e. the open string metric and coupling constant, as well as the zero-force condition are shown to be invariant under the $\mathrm{O}(p+1,p+1)$ transformation, i.e. deformation independent. We find sufficient conditions, including zero force and constant dilaton in the $ISO(3,1)$-invariant D3-brane solution, for exact S-duality between noncommutative Yang-Mills and open string theories. These results are used to construct noncommutative field and string theories with $\mathcal{N}=1$ supersymmetry from the $T^{(1,1)}$ and Pilch-Warner solutions. The latter has a non-trivial zero-force condition due to the warping.

Keywords: Duality in Gauge Field Theories, D-branes, String Duality, Superstring Vacua.

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1. Introduction

String theory has provided a surprising connection between gauge theories and supergravity. After taking certain decoupling limits one can learn a great deal about the large N limit of super-Yang-Mills by studying its dual supergravity description [1, 2]. This approach of using the supergravity dual to obtain a description of the large N limit of the theory has also been applied in many other circumstances. In particular, it has been applied to the study of noncommutative Yang-Mills theories [3, 4] as well as field theories with fewer supersymmetries and RG-flows [5]–[9].

A remarkable recent development has been the investigation of the S-dual description of $\mathcal{N} = 4$ noncommutative super-Yang-Mills (a D3-brane with background Neveu-Schwarz two-form potential) which is provided by a noncommutative open string (NCOS) theory [10, 11] on the three brane. In fact, one can construct an NCOS theory on any Dp-brane for $p \leq 5$. It is possible to view these NCOS theories as ultraviolet completions of the Yang-Mills theories with near critical background electric fields (though for $p = 5$ there is strong evidence that the theory actually contains also a (closed) little string sector [12]–[14]). Taking the strong coupling limit of the noncommutative open string theory on the D4-brane leads to the so called OM theory which in some sense acts as the M-theory for the open string theory [12, 15, 16]. This is a five-brane in a background with constant critical three-form potential, see also [17, 18]. The supergravity duals of these open string and membrane theories are known [19]–[26]. One may then use these dual supergravity solutions to determine many non-trivial properties of the NCOS and OM theories [22, 23, 25, 26]. In particular, the thermodynamics has been investigated with non-extremal versions of these solutions [4, 23].

In section 2 we first review the decoupling limits that lead to noncommutative theories on a D-brane and discuss the ultraviolet limit of the near-horizon region of various supergravity backgrounds.

In section 3 we then use the supergravity dual picture to address the issue of whether the noncommutativity parameter Θ can be tuned independently, or if a shift in Θ induces a variation of the other moduli. To examine this, we first give a simple form of Dp-brane bound states involving fluxes parametrized by an $\mathrm{O}(p+1, p+1)$ group element, as discussed in [28]. We then go to the near-horizon limit and show how the decoupling limits can be obtained as UV limits of the supergravity backgrounds. Using these general bound states we find that there is no deformation of the open string metric and coupling constant, and that the probe brane potential is deformed by a multiplicative factor hence preserving the zero-force condition. Actually, the latter result will be crucial for the discussion of S-duality in section 5.

We then exemplify these results in section 4 using some simple models, namely the maximally symmetric extremal Dp-branes, the $T^{(1,1)}$ extremal D3-brane and the D1–D5-brane system.

In section 5 we extend the deformation procedure to cases where one starts from the near-horizon solution (instead of the full brane solution). For D3-branes, we give a set of sufficient criteria for when the *critical electric* solution can be obtained from the magnetically deformed near-horizon solution by type-IIB S-duality (since the former cannot be directly obtained by an electric $\mathrm{O}(p+1, p+1)$ transformation). In particular, one condition is that the configuration is restricted to the subspace where the probe brane potential vanishes. This is trivially satisfied for maximally supersymmetric Dp-branes.

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The construction in section 5 is useful since it does not require the full brane solution, and in section 6 we use this setup to construct an $\mathcal{N}=1$ noncommutative Yang-Mills theory and its S-dual string theory based on the Pilch-Warner solution [7]. This theory has a non-trivial probe brane potential, and therefore illustrates the crucial importance for S-duality played by the condition of vanishing potential.

2. Open strings and supergravity duals

The data governing the effective open string perturbation theory on a Dp-brane in a closed string background with string frame metric $g_{MN} = g_{\mu\nu} \oplus g_{ij}$, dilaton e^{ϕ} and two-form potential B_{MN} is given by the open string two-point functions [29][1] together with the effective open string coupling. In this paper we use ten-dimensional spacetime indices $M = 0, \ldots, 9$, $(p+1)$-dimensional world volume indices $\mu = 0, \ldots, p$ and $(9-p)$-dimensional transverse space indices $i = p+1, \ldots, 9$. The two-point function is

$$\begin{aligned} \langle X^\mu(0) X^\nu(\tau)\rangle &= -\alpha' G^{\mu\nu}\log\tau + i\pi\Theta^{\mu\nu}\epsilon(\tau)\,, \\ \langle X^i(0) X^j(\tau)\rangle &= -\alpha' g^{ij}\log\tau\,, \end{aligned} \tag{2.1}$$

where

$$\alpha' G^{\mu\nu} + \Theta^{\mu\nu} = \alpha'\left(\frac{1}{g + 2\pi\alpha'\langle\mathcal{F}\rangle}\right)^{\mu\nu}\,, \tag{2.2}$$

and the effective open string coupling is

$$G_{\mathrm{O}} = e^{\phi}\sqrt{\frac{\det(g + 2\pi\alpha'\langle\mathcal{F}\rangle)}{\det g}}\,. \tag{2.3}$$

Here $\langle\mathcal{F}\rangle$ is the background value of the gauge invariant field strength on the Dp-brane:

$$\mathcal{F} = dA + \frac{1}{2\pi\alpha'}(f^*B)\,, \tag{2.4}$$

that is

$$\langle\mathcal{F}\rangle_{\mu\nu} = \frac{1}{2\pi\alpha'}(f^*B)_{\mu\nu}\,. \tag{2.5}$$

The symmetric part of (2.2), $G^{\mu\nu}$, is interpreted as the open string cometric, its inverse being the open string metric. The antisymmetric part $\Theta^{\mu\nu}$ is the noncommutativity parameter. Thus the open string endpoints see an effective D-brane world volume with metric

$$ds^2(G) = G_{\mu\nu}dx^\mu dx^\nu\,, \tag{2.6}$$

[1] Our definition of the noncommutativity parameter Θ differs from the one in ref. [29] by a factor 2π. The present normalization yields exact equality between Θ and the deformation parameter θ in section 3.

and deformed algebra of functions with star-product based on the Poisson structure

$$\Theta = \Theta^{\mu\nu}\partial_\mu\partial'_\nu\,. \tag{2.7}$$

We note the following useful identities:

$$G_{\mu\nu} = g_{\mu\nu} - (f^*B)_{\mu\rho}g^{\rho\sigma}(f^*B)_{\sigma\nu}\,, \tag{2.8}$$

$$G_{\rm O} = e^\phi\left(\frac{\det G}{\det g}\right)^{1/4}, \qquad \Theta^{\mu\nu} = -\alpha' g^{\mu\rho}(f^*B)_{\rho\sigma}G^{\sigma\nu}\,. \tag{2.9}$$

We are interested in examining a Dp-brane probe in a background with large Dp-brane charge. For large charge the background fields are slowly varying functions of the distance between the probe brane and the stack of source branes. The size of open string fluctuations around the probe are governed by the effective open string coupling $G_{\rm O}$ [10] (while the size of closed string fluctuations in the bulk are governed by the closed string coupling e^ϕ). In a region where $G_{\rm O} \ll 1$ we have a one-parameter family of open string quantum theories on the probe brane defined by the open string data given above and two energy scales: the mass of the higgsed W bosons, above which the interactions between the probe and the stack of source branes no longer is negligible, and the effective Planck energy, which sets the cutoff for the closed string sector.

A fruitful feature of this setup is the possibility under certain conditions to consider a near-horizon region of the background, which defines a supergravity dual,[2] where the one-parameter family extends all the way to the extreme UV. This amounts to taking the separation (given in rescaled units in the near-horizon region) between the probe brane and the source branes to infinity as to decouple both W bosons and closed strings while keeping a finite (and small) open string coupling. One important condition for the UV-completion to be physically well-defined is that it should have a stable ground state, i.e. the zero-force condition should extend into the UV. For finite separation it describes the flow from the UV-completion down to the gauge theory in the extreme IR.

There are several important reasons why we prefer this holographic setup in favour of simply scaling 'flat space' closed string moduli. Firstly, it is not always the case that the total system becomes weakly coupled so that it makes sense to describe the limits in a flat background. The advantage of the holographic setup is then of course that the non-perturbative aspects are automatically included in the description of the limit. In particular, the relations between the critical scalings of the various field strengths and tensions appear naturally in the bound state solutions.

[2]To avoid confusion, we wish to clarify that we shall distinguish between the full brane solution and the near-horizon region and we shall always refer to the latter as the supergravity dual, even though this term could of course also be used for the full brane solution. Note also that the near-horizon limit in the NCOS case is defined in a slightly different manner from the standard one; this fact is discussed in more detail in section 3.2.

Moreover, a bound state solution typically contains both magnetic *and* dual electric fluxes that couple to dual open branes. For instance, as we shall see in section 3, a magnetic NS flux typically comes with a 'crossed' electric RR flux and vice versa. Hence, instead of considering different source branes with electric or magnetic fluxes, one may for any one given source brane obtain dual descriptions of the theory on the probe brane simply by switching from one open string (e.g. from an F1 to a D1) or brane description to another. A third, perhaps less profound but nevertheless practical reason is that due to the 'back-reaction' in the bound state solution from switching on the noncommutative deformation parameter, the various issues raised recently of whether and how the deformation acts on the other moduli in the theory can be answered in a both precise and concrete fashion.

Of the various dual open branes[3] referred to above, the two cases which are presently best understood are when the decoupled (and sometimes complete) UV-limits turn out to be a noncommutative gauge field theory (NCYM) or an open string theory (NCOS). A NCYM arises when the open string metric (2.6) diverges in units of α', while the NCOS arises in case this quantity is fixed. The basic reason is that the rest-mass of an open string state with oscillator number $N \geq 1$ is proportional to $\sqrt{|G_{00}|(N-1)/\alpha'}$. Actually, the two possibilities are sensitive to the signature of the noncommutativity parameter: NCYM requires magnetic $\Theta^{\mu\nu}$ and NCOS electric $\Theta^{\mu\nu}$. This relates to the fact that the asymptotic geometry of the background is that of an array of smeared F1-strings for NCOS and smeared D$(p-r)$-branes for NCYM with r the rank of the tensor field generating the deformation.

Let us consider the case of a rank 2 background two-form potential $B_{\mu\nu}$ and let the limit, in which the separation between the probe and the sources diverges, be controlled by some parameter $\epsilon \to 0$. Suppose the closed string data in the brane directions obey the following asymptotic scaling behaviour in the near-horizon region [29]–[31] (for NCYM $\alpha, \beta = 0, \ldots, p-2$ and $a, b = p-1, p$; for NCOS $\alpha, \beta = 0, 1$ and $a, b = 2, \ldots, p$; $\lambda > 0$):

$$\begin{aligned}
\text{NCYM}: \quad & \frac{g_{\alpha\beta}}{\alpha'} \sim \eta_{\alpha\beta}\epsilon^{-\lambda}\,, \qquad \frac{B_{\alpha\beta}}{\alpha'} = 0\,, \\
& \frac{g_{ab}}{\alpha'} \sim \delta_{ab}\epsilon^{\lambda}\,, \qquad \frac{B_{ab}}{\alpha'} \sim \epsilon_{ab}\epsilon^{0}\,, \\
& e^{\phi} \sim \epsilon^{\frac{1}{2}(5-p)\lambda}\,. \qquad\qquad (2.10)
\end{aligned}$$

$$\begin{aligned}
\text{NCOS}: \quad & \frac{g_{\alpha\beta}}{\alpha'} \sim \eta_{\alpha\beta}\epsilon^{-\lambda}(1 + U\epsilon^{\lambda} + \cdots)\,, \\
& \frac{B_{\alpha\beta}}{\alpha'} \sim \epsilon_{\alpha\beta}\epsilon^{-\lambda}(1 + V\epsilon^{\lambda} + \cdots)\,, \qquad U + V \neq 0\,, \\
& \frac{g_{ab}}{\alpha'} \sim \delta_{ab}\epsilon^{0}\,, \qquad \frac{B_{ab}}{\alpha'} = 0\,, \\
& e^{\phi} \sim \epsilon^{-\frac{1}{2}\lambda}\,. \qquad\qquad (2.11)
\end{aligned}$$

[3]These include besides the open strings and OM cases also the ODp cases in [12, 14].

We remark that the physics depends on α' only via the tension $g_{\mu\nu}/\alpha'$ and $B_{\mu\nu}/\alpha'$; thus the above limits can be considered either as $\alpha' \to 0$ limits in a fixed, flat background or as UV-limits in a supergravity dual i.e. $r \to \infty$, where the tension does not scale with α', its scaling instead being determined by the energy scale set by the distance between the probe and the source branes.

Both (2.10) and (2.11) yield infinite energies for massive closed string states, whose rest-mass scales as $\sqrt{|g_{00}|/\alpha'}$, while the energies of massive open string states diverges for (2.10) and remains finite for (2.11), as explained above. A more careful analysis [32] based on calculations of absorption cross-sections shows, however, that the massless closed string sector only decouples for $p \leq 5$. In the case of $p = 5$, i.e. the type-IIB D5-brane, there is also compelling evidence that the theory also contains a (closed) little string sector [14].

Next we are going to describe a construction of Dp-brane bound states with maximal rank deformations, and we shall then discuss the rank 2 case using the above limits.

3. Electric and magnetic deformations of open string data

We begin by describing how one may in a very efficient way construct the relevant deformed background given an undeformed brane solution. We then explain how the near-horizon limit of the deformed solution is obtained. This is followed by the probing of the new background solution via a D-brane probe. Finally, we analyse the open string data given in the previous section on the probe D-brane in this background.

3.1 Dp-brane bound states from $\mathrm{O}(p+1, p+1)$ transformations

Many examples of Dp-brane bound states have been constructed in the literature, see e.g. [19, 20, 33, 34]. The basic construction method is to combine a series of diagonal T-dualities [35, 36], constant NS gauge transformations[4] and $SO(p,1)$ transformations, which together make up the T-duality group $\mathrm{O}(p+1, p+1)$. Recently, it was found that under a general transformation, the RR potentials actually transform in a chiral $Spin(p+1, p+1)$ representation [37, 38], and in [28] this was used to give a corresponding general parametrization of Dp-brane bound states as follows. Given a supergravity solution, one first T-dualizes in the directions where one wants to turn on NS fluxes, and then one shifts B_2 with a constant in these directions. After this one T-dualizes back again. In a more concise language, the deformation with parameter $\theta^{\mu\nu}/\alpha'$ is generated by the following $\mathrm{O}(p+1, p+1)$ T-duality group

[4] There is an alternative method using a boost/rotation between a compact and a non-compact direction instead of a gauge transformation. This method gives equivalent results to the method used in this paper and will not be discussed any further.

element[5]

$$\Lambda = \Lambda_0 \ldots \Lambda_p \Lambda_{\theta/\alpha'} \Lambda_p \cdots \Lambda_0 = J \Lambda_{\theta/\alpha'} J = \Lambda^T_{-\theta/\alpha'} = \begin{pmatrix} 1 & 0 \\ \theta/\alpha' & 1 \end{pmatrix}. \tag{3.1}$$

Here $\theta^{\mu\nu}$ has dimension (length)^2 and carries indices upstairs since it starts life on the T-dual world volume. In the NS-NS sector an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ transforms $E = g+B$ by means of a projective transformation, i.e. $\tilde{E} = (aE + b)(cE + d)^{-1}$. In the RR sector the anti-symmetric tensor fields can be shown to correspond to chiral spinors of the T-duality group. This follows from mapping the inner product i_μ and one-form dx^μ onto the annihilation and creation operators that can be formed by taking linear combinations of the relevant Dirac matrices. The result of this construction on the anti-symmetric tensor fields is given below. We shall assume that the undeformed brane configuration (in the string frame) is given by :

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu + g_{ij} dx^i dx^j \,, \qquad B_2 = \beta_2 \,, \\ C &= \omega dx^0 \wedge \cdots dx^p \wedge (1+\alpha) + \gamma \,, \end{aligned} \tag{3.2}$$

where C, and in particular γ, is a generalized sum of forms of different degree, and β_2, γ and α are transverse forms, i.e.

$$i_\mu \beta_2 = i_\mu \alpha = i_\mu \gamma = 0 \,. \tag{3.3}$$

Here i_μ denotes inner product with the vector field associated with x^μ. The deformed configuration is determined as follows [37]:

$$\tilde{g}_{\mu\nu} + \tilde{B}_{\mu\nu} = \left[(g^{-1} + \theta/\alpha')^{-1} \right]_{\mu\nu} \,, \quad e^{\tilde{\phi}} = e^{\phi} \left(\frac{\det \tilde{g}}{\det g} \right)^{1/4} \,, \tag{3.4}$$

$$\tilde{g}_{\mu i} = \tilde{B}_{\mu i} = 0 \,, \qquad \tilde{g}_{ij} = g_{ij} \,, \qquad \tilde{B}_{ij} = \beta_{ij} \,, \tag{3.5}$$

$$\tilde{C} = \omega \left[\exp(-b_2) \exp\left(\frac{1}{2\alpha'} \theta^{\mu\nu} i_\mu i_\nu \right) dx^0 \wedge \cdots dx^p \right] \wedge (1+\alpha) + \exp(-b_2) \wedge \gamma \,, \tag{3.6}$$

where b_2 is the deformed two-form in the brane directions. In these directions we have

$$d\tilde{s}^2_{p+1} = \left[g \left(1 - (\theta g/\alpha')^2 \right)^{-1} \right]_{\mu\nu} dx^\mu dx^\nu \,, \tag{3.7}$$

$$b_2 = -\frac{1}{2\alpha'} \left[g\theta g \left(1 - (\theta g/\alpha')^2 \right)^{-1} \right]_{\mu\nu} dx^\mu \wedge dx^\nu \,. \tag{3.8}$$

In a frame where the metric and $(\theta g)^\mu{}_\nu$ are block-diagonal we find that in the k'th 2×2 block

$$d\tilde{s}^2_{(k)} = \frac{ds^2_{(k)}}{h_k} \,, \quad b_2^{(k)} = -\frac{\theta_k \det{}_{(k)} g}{2\alpha' h_k} \epsilon_{\mu\nu} dx^\mu_{(k)} \wedge dx^\nu_{(k)} \,, \quad h_k = 1 + \left(\frac{\theta_k}{\alpha'} \right)^2 \det{}_{(k)} g \,, \tag{3.9}$$

[5]See [37, 28] for conventions and definitions of the various elements of $\mathrm{O}(p+1, p+1)$ appearing in the following discussion.

and that the dilaton is given by

$$e^{2\tilde{\phi}} = \frac{e^{2\phi}}{\prod_k h_k} \,. \tag{3.10}$$

We assume that the matrix $\delta^\mu_\nu - (\alpha')^{-2}[(\theta g)^2]^\mu{}_\nu$ is invertible (for all $g_{\mu\nu}$), such that the configuration in (3.7), (3.8) is non-singular. In the special case of a rank 2 deformation we get

$$d\tilde{s}^2 = \frac{ds'^2}{h} + ds_8^2 \,, \quad \tilde{B}_2 = -\frac{\theta \det{}' g}{2\alpha' h}\epsilon_{\mu\nu} dx'^\mu \wedge dx'^\nu + \beta_2 \,, \tag{3.11}$$

$$e^{2\tilde{\phi}} = \frac{e^{2\phi}}{h} \,, \quad h = 1 + \left(\frac{\theta}{\alpha'}\right)^2 \det{}' g \,, \tag{3.12}$$

$$\begin{aligned}\tilde{C} = \omega &\left[\frac{1}{h} dx^0 \wedge \cdots dx^p + \frac{\theta}{2\alpha'}\epsilon^{\mu\nu} i'_\mu i'_\nu dx^0 \wedge \cdots \wedge dx^p\right] \wedge (1+\alpha) + \\ &+ \left[1 + \frac{\theta \det{}' g}{2\alpha' h}\epsilon_{\mu\nu} dx'^\mu \wedge dx'^\nu\right] \wedge \gamma \,,\end{aligned} \tag{3.13}$$

where x'^μ denote the two directions where the deformation is non-trivial and $\det{}'$ is the 2×2 determinant in this space.

It is worth mentioning that formally, the $\mathrm{O}(p+1, p+1)$ transformations of the solutions require the brane solution to be wrapped on a $(p+1)$-dimensional torus, which may be decompactified after the transformation. However, keeping compact directions, the Dp-brane bound states give rise to supergravity duals of the wrapped noncommutative open string theories of [39].

3.2 Near-horizon regions for deformed extremal branes

For an extremal brane solution the metric in the brane directions can be written as

$$g_{\mu\nu} = \mathcal{E}^2 \eta_{\mu\nu} \,, \tag{3.14}$$

where $\mathcal{E}$ is a function of the transverse coordinates that interpolates between the near-horizon region of the brane and an asymptotical region such that

$$0 < \mathcal{E} < 1 \,. \tag{3.15}$$

In the case of a rank 2 magnetic deformation, that is $\det{}' g > 0$, the near-horizon region is defined in the same way as the one of the undeformed configuration, i.e. the supergravity dual of the (commutative) field theory on the brane. Thus, as we take $\alpha' \to 0$ we keep fixed the following quantities:

$$\text{Magnetic near-horizon:} \quad \left.\begin{matrix} x^\mu \,, U^2 \equiv \dfrac{\mathcal{E}^2}{\alpha'} \,, \Phi^i \equiv \dfrac{x^i}{\alpha'} \,, \\ \theta \,, \quad g_{YM}^2 \equiv g(\alpha')^{\frac{p-3}{2}} \end{matrix}\right\} \text{fixed} \,. \tag{3.16}$$

Here it is important to note that the canonical dimensions of the fields are chosen such that if ds^2/α', B_2/α' and $C_p/(\alpha')^{p/2}$ are held fixed in the $\alpha' \to 0$ limit, then the supergravity lagrangian is finite. In the absence of other dimensionful parameters, in the $\alpha' \to 0$ limit it is U as a function of the vacuum expectation values of the scalar fields Φ^i, that sets the energy scale on the probe brane (it is assumed that x^i are coordinates of dimension length). In the UV limit, $U \to \infty$, one recognizes the NCYM limit (2.10) with the undeformed dilaton scaling as

$$e^{\phi} \sim g_{\rm YM}^2 U^{p-3} \,, \tag{3.17}$$

where $g_{\rm YM}^2$ is the Yang-Mills coupling in $p+1$ dimensions.

Note that the requirement that the scale is determined by U implies that the pull-backs of the transverse background components scales to zero in the UV. For example, the scalar kinetic term will scale as

$$f^*(g_{ij}dx^i dx^j) \sim \alpha' U^{-2} dx^\mu dx^\nu \partial_\mu \Phi^i \partial_\nu \Phi^j \bar{g}_{ij} \,, \tag{3.18}$$

where f denotes the embedding of the probe brane in spacetime. Here $\bar{g}_{ij}$ is a dimensionless function of transverse coordinates which does not scale in the UV and the scalar gradients $\partial_\mu \Phi^i$ are fixed in the UV.

We remark that in many cases the field theory near-horizon geometry is warped [40], such that the near-horizon geometry factorizes into that of a $(p+2)$-dimensional anti-de Sitter space and an $(8-p)$-dimensional internal space in such a way that the energy scale U defined in (3.16) and the energy scale u of the near horizon anti-de Sitter part of the spacetime metric are related as follows:

$$\begin{aligned} ds^2 &= \Omega^2 ds^2_{AdS_{p+2}} + d\hat{s}^2_{8-p} \,, \quad ds^2_{AdS_{p+2}} = u^2 dx^2_{p+1} + \frac{du^2}{u^2} \,, \\ U^2 &= \Omega^2 u^2 \,, \end{aligned} \tag{3.19}$$

where the warp factor Ω is a dimensionless (non-constant) function of the internal coordinates in the line element $d\hat{s}^2_{8-p}$. This is equivalent to the internal metric defined in (3.18) being block-diagonal as follows:

$$d\bar{s}^2_{9-p} = \Omega^4 (du^2 + u^2 d\hat{s}^2_{8-p}) \,. \tag{3.20}$$

As we shall see in the next subsection, this gives rise to a probe brane potential V. Excitations that break the zero-force condition have a potential energy that scales like $V \sim g_{\rm YM}^{-2} U^4 \to \infty$ in the UV, so they will be frozen out in taking the UV limit. In what follows it will be useful to define the map i as follows

$$i : V^{-1}(0) \mapsto M^{10} \,, \tag{3.21}$$

it is the embedding of the submanifold where the potential energy of a probe D-brane vanishes in the ten-dimensional spacetime M^{10}. This submanifold is called *the vanishing locus.*

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In the case of an electric deformation we have $\det{}' g < 0$, so the 'field theory' near-horizon limit (3.16) would yield a supergravity dual with a singularity at a finite energy scale (where h would vanish). To avoid this, the electric near-horizon region is instead defined by the following critical limit[6], as $\alpha' \to 0$:

$$\text{Electric near-horizon:} \quad \frac{x^M}{\sqrt{\alpha'}}\,,\ \mathcal{E}\,,\ g\,,\ \omega \text{ fixed}\,; \quad \frac{\theta}{\alpha'} \to 1\,. \tag{3.22}$$

Provided that there are no other dimensionful parameters than θ and α', then, since all coordinates are kept fixed in units of α', the electric near-horizon region contains the original asymptotic region. The UV limit now corresponds to $h = 1 - \mathcal{E}^4 \to 0$, which can be seen to reproduce the NCOS limit (2.11) (using (3.15)). We remark that the spacetime metric approaches that of a smeared string in the UV; taking the electric deformation in the 0 and 1 directions we have

$$\text{NCOS}: \quad \frac{ds^2}{\alpha'} \overset{\text{UV}}{\sim} h^{-1}(-dx_0^2 + dx_1^2) + dx_2^2 + \cdots + dx_p^2 + \bar{g}_{ij} dx^i dx^j\,, \tag{3.23}$$

where $\bar{g}_{ij}$ is defined in (3.18). It follows that all closed string states except the massless states with momentum along the string are frozen out from the perturbative spectrum in the sense that their energy diverges. Similarly, from (3.18) it follows that for the NCYM limit the asymptotic metric on the vanishing locus approaches that of a smeared D$(p-2)$-brane:

$$\text{NCYM}: \quad \frac{ds^2}{\alpha'} \overset{\text{UV}}{\sim} u^2(dx_0^2 + \cdots + dx_{p-2}^2) + u^{-2}(dx_{p-1}^2 + dx_p^2 + i^* d\Phi_{p+1}^2 + \cdots + i^* d\Phi_9^2)\,. \tag{3.24}$$

and the perturbative closed string spectrum reduces in the UV to the massless modes with momentum parallel to the smeared brane. Hence, at the perturbative level the dynamics of the Dp-brane is decoupled from the bulk.

The deformed open string data in the near-horizon region is found by inserting the deformed closed string configuration (3.4) into (2.8) and (2.9). For the open string metric and noncommutativity parameter we find

$$\tilde{G}_{\mu\nu} = g_{\mu\nu}\,, \quad \tilde{\Theta}^{\mu\nu} = \theta^{\mu\nu}\,. \tag{3.25}$$

Hence the open string metric is undeformed. Using (2.3) we then find that also the open string coupling is undeformed:

$$\tilde{G}_{\rm O} = e^{\tilde{\phi}} \left(\frac{\det \tilde{G}}{\det \tilde{g}}\right)^{1/4} = e^{\phi} \left(\frac{\det \tilde{g}}{\det g}\right)^{1/4} \left(\frac{\det g}{\det \tilde{g}}\right)^{1/4} = e^{\phi}\,. \tag{3.26}$$

Actually, starting from a more general brane configuration in which there are already NS fluxes inside the world volume one still finds that

$$\tilde{G}_{\mu\nu} = G_{\mu\nu}\,, \qquad \tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \theta^{\mu\nu}\,, \qquad \tilde{G}_{\rm O} = G_{\rm O}\,. \tag{3.27}$$

[6] g is here the closed string coupling constant.

We also see that the noncommutativity parameter must remain constant along the RG-flow.[7] One might expect the noncommutativity to vanish in the IR because the Neveu-Schwarz two-form vanishes. This is however not the case due to the compensating scaling of the metric. A further understanding of this would be desirable, even though it seems that the effects of the constant noncommutativity parameter vanishes in the IR, since fields become slowly varying in this limit.

We remark that for an NCOS theory, it follows from (3.22) that in the UV the open string tension and noncommutativity scale as

$$\frac{ds^2(\tilde{G})}{\alpha'} \sim \frac{\eta_{\mu\nu}dx^\mu dx^\nu}{\alpha'}, \qquad \tilde{\Theta} \sim \alpha' \epsilon^{\alpha\beta}\partial_\alpha \partial^{\prime}_\beta \,, \tag{3.28}$$

such that indeed the inverse of the effective open string tension is equal to the noncommutativity parameter (after going to fixed coordinates, the coordinates in (3.28) obey (3.22)).

3.3 Non-deformation of zero-force condition

It was shown in [28] that the zero-force condition for the probe brane in the supergravity dual is not deformed by a noncommutative deformation. To examine this, one uses the UV expansion of the Born-Infeld lagrangian, which is discussed in appendix A (see (A.20)), and that the contribution to the potential from the WZ term is given by

$$\begin{aligned}\tilde{\mathcal{L}}_{\rm WZ} &= \omega dx^0 \wedge \cdots \wedge dx^p \wedge \exp\left(\frac{\theta^{\mu\nu}}{2\alpha'} i_\mu i_\nu\right) \exp\left(\frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu\right) \\ &= \omega\sqrt{\det(1+\theta F)}dx^0 \wedge \cdots \wedge dx^p \,,\end{aligned} \tag{3.29}$$

where the bound state solution (3.11) and (3.13) have been used. Note that the determinant is a square of pfaffians, so that the last expression is indeed a finite polynomial in traces of θF. The deformed potential is therefore given by

$$\tilde{V} = \sqrt{\det(1+\theta F)}\; V \,, \tag{3.30}$$

where

$$V = (\alpha')^{-\frac{p+1}{2}}\left(e^{-\phi}\sqrt{-\det g_{\mu\nu}} - \omega\right) \tag{3.31}$$

is the undeformed potential, which shows that the zero-force condition is not deformed by turning on a noncommutativity parameter.

4. Three examples

In this section we give three examples where the warp factor Ω in (3.19) is constant, and the probe brane potential is identically zero. A case with non-trivial warp factor will be examined in section 6.

[7] This was observed using the explicit supergravity solution in [26], and deduced from a nonrenormalization argument using supersymmetry already in [27].

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4.1 Stack of maximally symmetric extremal Dp-branes

As the canonical example we consider a stack of maximally symmetric extremal Dp-branes ($p \leq 6$):

$$
\begin{aligned}
ds^2 &= H^{-1/2} dx^2 + H^{1/2}(dr^2 + r^2 d\Omega^2)\,, \\
e^{2\phi} &= g^2 H^{\frac{3-p}{2}}\,, \qquad H = 1 + \frac{gN(\alpha')^{\frac{7-p}{2}}}{r^{7-p}}\,, \\
C &= \frac{1}{gH} dx^0 \wedge \cdots \wedge dx^p + (7-p)N(\alpha')^{\frac{7-p}{2}} \epsilon_{7-p}\,,
\end{aligned}
\tag{4.1}
$$

where $d\epsilon_{7-p}$ is the volume element on the transverse $(8-p)$-sphere. After a rank 2 deformation, using the T-duality transformation described above, the magnetically deformed configuration is

$$
\begin{aligned}
d\tilde{s}^2 &= H^{-1/2}\left(-dx_0^2 + \ldots + dx_{p-2}^2 + \frac{1}{h}(dx_{p-1}^2 + dx_p^2)\right) + H^{1/2}(dr^2 + r^2 d\Omega^2)\,, \\
e^{2\tilde{\phi}} &= \frac{1}{h} g^2 H^{\frac{3-p}{2}}\,, \\
\tilde{B} &= -\frac{\theta^{p-1,p} H^{-1}}{\alpha' h} dx^{p-1} \wedge dx^p\,, \\
\tilde{C} &= \frac{1}{gHh} dx^0 \wedge \cdots \wedge dx^p + (7-p)N(\alpha')^{\frac{7-p}{2}}(1-\tilde{B}) \wedge \epsilon_{7-p} - \\
&\quad - \frac{\theta^{p-1,p}}{gH\alpha'} dx^0 \wedge \cdots \wedge dx^{p-2}\,, \\
\text{where} \quad h &\equiv 1 + \left(\frac{\theta^{p-1,p}}{\alpha'}\right)^2 H^{-1}\,,
\end{aligned}
\tag{4.2}
$$

and the electrically deformed one is

$$
\begin{aligned}
d\tilde{s}^2 &= H^{-1/2}\left(\frac{1}{h}(-dx_0^2 + dx_1^2) + dx_2^2 + \ldots + dx_p^2\right) + H^{1/2}(dr^2 + r^2 d\Omega^2)\,, \\
e^{2\tilde{\phi}} &= \frac{1}{h} g^2 H^{\frac{3-p}{2}}\,, \\
\tilde{B} &= \frac{\theta^{01} H^{-1}}{\alpha' h} dx^0 \wedge dx^1\,, \\
\tilde{C} &= \frac{1}{gHh} dx^0 \wedge \cdots \wedge dx^p + (7-p)N(\alpha')^{\frac{7-p}{2}}(1-\tilde{B}) \wedge \epsilon_{7-p} \\
&\quad - \frac{\theta^{01}}{gH\alpha'} dx^2 \wedge \ldots \wedge dx^p\,, \\
\text{where} \quad h &\equiv 1 - \left(\frac{\theta^{01}}{\alpha'}\right)^2 H^{-1}\,.
\end{aligned}
\tag{4.3}
$$

We define the magnetic and electric supergravity duals by the resulting near-horizon limits, taking $\alpha' \to 0$,

$$\text{Magnetic near-horizon}: \quad x^\mu,\ \frac{r}{\alpha'},\ \theta,\ g^2_{\rm YM} \equiv g(\alpha')^{\frac{p-3}{2}} \quad \text{fixed}. \tag{4.4}$$

$$\text{Electric near-horizon}: \quad \frac{x^\mu}{\sqrt{\alpha'}},\ \frac{r}{\sqrt{\alpha'}},\ g \quad \text{fixed}; \quad \frac{\theta}{\alpha'} \to 1. \tag{4.5}$$

These limits are easily seen to arise from the brane solutions by first introducing the relevant rescaled radial coordinate in the two, i.e. electric and magnetic, cases. The other definitions in (4.4) and (4.5) then follow by demanding that the quotients of the background fields and powers of α' defined in section 3.1 should be independent of α'. The near-horizon geometry thus obtained interpolates between conformal $AdS_{p+2} \times S^{8-p}$ and an array of smeared D$(p-2)$-branes for NCYM and F1-strings for NCOS. We verify (2.10) and (2.11) as follows:

$$\text{NCYM}: \epsilon = H\ , \lambda = \frac{1}{2}. \tag{4.6}$$

$$\text{NCOS}: \epsilon = h\ , \lambda = 1\ , U = -\frac{1}{2}\ , V = -1. \tag{4.7}$$

Note that for NCYM it follows from (4.4) that $(\alpha')^2 H$ is fixed, i.e. independent of α', in the near-horizon region, where it vanishes in the UV, so that strictly speaking we should take $\epsilon = (\alpha')^2 H/\ell^2$ in (4.6) for some fixed length scale ℓ. We also remark that (3.17) is indeed obeyed with the definition of the Yang-Mills coupling made in (4.4). Using (2.8) and (2.9) we compute the open string data in the complete brane configuration as follows:

$$\tilde{G}_{\mu\nu} = H^{-1/2}\eta_{\mu\nu}\,, \quad \tilde{G}_{\rm O} = gH^{\frac{3-p}{4}}\,, \quad \tilde{\Theta}^{\mu\nu} = \theta^{\mu\nu}\,, \tag{4.8}$$

where $\theta^{\mu\nu}$ is magnetic or electric as the case may be. From (4.4) and (4.5) it follows that in the near horizon region the following open string data are fixed:

$$\frac{ds^2(\tilde{G})}{\alpha'} = \frac{H^{-\frac{1}{2}}dx^2}{\alpha'}\,, \quad \tilde{G}_{\rm O} = gH^{\frac{3-p}{4}}\,, \quad \tilde{\Theta} = \theta^{\mu\nu}\partial_\mu\partial'_\nu\,. \tag{4.9}$$

For NCOS, it follows from (4.5) that the above quantities are indeed finite in the UV limit (4.7). By going to fixed coordinates, which requires the introduction of some arbitrary length scale $\sqrt{\alpha'_{\rm eff}}$, it follows that the inverse of the open string tension and the spatio-temporal noncommutativity parameter are indeed equal and given by $\alpha'_{\rm eff}$ in the UV.

For NCYM, it follows from (4.4) that in the UV limit (4.6) the noncommutativity $\tilde{\Theta}$ is finite, while the open string tension $ds^2(\tilde{G})/\alpha'$ diverges. However, defining a fixed metric $ds^2(\bar{G})$ and length scale ℓ by

$$\text{NCYM}\ :\ \frac{ds^2(\tilde{G})}{\alpha'} \sim \epsilon^{-\frac{1}{2}}\frac{ds^2(\bar{G})}{\ell^2}\,; \quad ds^2(\bar{G})\,, \ell \quad \text{fixed}\,, \tag{4.10}$$

yields a finite kinetic term in the Born-Infeld lagrangian of the probe D-brane:

$$S_{\rm DBI} = -\frac{1}{4g_{\rm YM}^2}\int d^{p+1}x\sqrt{-\det\bar{G}}\bar{G}^{\mu\rho}\bar{G}^{\nu\sigma}F_{\mu\rho}F_{\nu\sigma}+\cdots\,, \tag{4.11}$$

$$g_{YM}^2 = \epsilon^{\frac{p-3}{4}}\ell^{p-3}\tilde{G}_{\rm O}\,, \tag{4.12}$$

where the fixed Yang-Mills coupling $g_{\rm YM}$ can be seen to agree with the definition (4.4). The ellipses in (4.11) represent more terms that are finite in the UV, while the leading divergent terms cancel against the WZ-term due to the zero-force condition, as explained in section 3.2 and 3.3. From the discussion in appendix A, it follows that the complete probe brane lagrangian, including the WZ-term, can be written as the noncommutative lagrangian (A.17) in the extreme UV region, where we can take

$$\bar{G}_{\mu\nu} = \eta_{\mu\nu}\,,\quad \bar{g}_{ij} = \delta_{ij}\,. \tag{4.13}$$

As discussed in section 2, it has important consequences for the interpretation of the resulting NCYM, that the open string coupling diverges in the UV for $p > 3$ and vanishes for $p = 2$, as follows from (4.12). For $p = 3$, the NCYM is a UV complete theory since the open string coupling is fixed in the UV. For $p = 4$, the noncommutative lagrangian (A.17) should be thought of as an effective field theory description valid at energies below $g_{\rm YM}^{-2}$. The bound state has a critical electric RR three-form. The UV completion is therefore an open D2-brane theory in five dimensions. By lifting this theory to M-theory along a magnetic circle of radius $g_{\rm YM}^2$ one finds that it is dual to the noncommutative M-theory five-brane [12, 15]. The five brane KK modes appears in the D4-brane NCYM theory as noncommutative instantons with energy $g_{\rm YM}^{-2}$. For $p = 5$ the bound state has a critical electric RR four-form, and the UV completion involves an open D3-brane.

4.2 The $T^{(1,1)}$ solution

We next consider an extremal D3-brane with less supersymmetry. The $T^{(1,1)}$ $\mathcal{N}$=1 brane solution is [41]:

$$\begin{aligned} ds^2 &= H^{-1/2}dx^2 + H^{1/2}(dr^2 + r^2 d\Omega_T^2)\,,\\ e^{2\phi} &= g^2\,,\qquad H = 1+\frac{gN(\alpha')^2}{r^4}\,. \end{aligned} \tag{4.14}$$

Here $d\Omega_T$ is the line element of $T^{(1,1)}$, which implies that the near-horizon region is $AdS_5 \times T^{(1,1)}$. This near-horizon region is the AdS/CFT dual of an $\mathcal{N}$=1 superconformal field theory [42]. Comparing this metric with (4.1) we see that they are identical except for the five-sphere which is changed to $T^{(1,1)}$.[8] If we now deform

[8] They also have the same C_{0123} component but not the same components of C_4 in the transverse directions.

this brane solution electrically (magnetically) and then take the NCOS (NCYM) limit, we will obtain the same open string metric, coupling constant and $\tilde{\Theta}$ as in the previous subsection. Because we start with a solution which gives identically vanishing potential for a probe brane, we therefore know that the $\mathcal{N}$=1 NCOS and NCYM theories have to be S-dual.[9]

We see here that starting with a D3-brane which has a near-horizon geometry which is a Freund-Rubin compactification of type-IIB supergravity is relatively easy to deform and investigate, because of its similarity with the maximally supersymmetric D3-brane.

4.3 Electric deformation of D1–D5 system

The D1–D5 system (with lorentzian signature) is not possible to deform magnetically in order to obtain a 1+1 dimensional NCYM theory. This is obvious since turning on a B-field in the D-string directions gives an electric deformation. To obtain a NCYM theory one has to start with a D1–D5-brane configuration with euclidean signature. This was done in [3].

To give the electrically deformed D1–D5 solution we start with the undeformed D1–D5 solution:

$$\begin{aligned}
ds^2 &= (H_1H_5)^{-1/2}dx^2 + H_1^{1/2}H_5^{-1/2}dy^2 + (H_1H_5)^{1/2}(dr^2 + r^2 d\Omega_3^2)\,,\\
e^{2\phi} &= g^2\frac{H_1}{H_5}\,, \qquad H_{1,5} = 1 + \frac{\alpha' g R_{1,5}^2}{r^2}\,,\\
C_2 &= \frac{1}{gH_1}dx^0\wedge dx^1 + 2\alpha' R_5^2\epsilon_2\,,\\
C_6 &= \left[\frac{1}{gH_5}dx^0\wedge dx^1 + 2\alpha' R_1^2\epsilon_2\right]\wedge dy^1\wedge dy^2\wedge dy^3\wedge dy^4\,,
\end{aligned}\tag{4.15}$$

where $d\Omega_3^2$ and $d\epsilon_2$ are the metric and the volume element on the transverse S^3. Taking $\theta^{\mu\nu}$ to be non-zero in the $0,1$-directions in (3.11)–(3.13) yields the following electrically deformed solution:

$$\begin{aligned}
d\tilde{s}^2 &= (H_1H_5)^{-1/2}\frac{1}{h}dx^2 + H_1^{1/2}H_5^{-1/2}dy^2 + (H_1H_5)^{1/2}(dr^2 + r^2 d\Omega_3^2)\,,\\
e^{2\tilde{\phi}} &= g^2\frac{H_1}{H_5 h}\,, \qquad h = 1 - \left(\frac{\theta}{\alpha'}\right)^2 (H_1H_5)^{-1}\,,\\
\tilde{B}_2 &= \frac{\theta}{\alpha' H_1 H_5 h}dx^0\wedge dx^1\,,\\
\tilde{C}_0 &= -\frac{\theta}{\alpha' g H_1}\,,\\
\tilde{C}_2 &= \frac{1}{gH_1 h}dx^0\wedge dx^1 + 2\alpha' R_5^2\epsilon_2\,,
\end{aligned}$$

[9] See section 5 for a discussion about exact S-duality.

$$\tilde{C}_4 = -\frac{\theta}{\alpha'}\left(\frac{1}{gH_5}dy^1\wedge dy^2\wedge dy^3\wedge dy^4 + \frac{2\alpha' R_5^2}{H_1H_5h}dx^0\wedge dx^1\wedge\epsilon_2\right),$$
$$\tilde{C}_6 = \left(\frac{1}{gH_5h}dx^0\wedge dx^1 + 2\alpha' R_1^2\epsilon_2\right)\wedge dy^1\wedge dy^2\wedge dy^3\wedge dy^4,$$
$$\tilde{C}_8 = -\frac{\theta}{\alpha' gH_1H_5h}dx^0\wedge dx^1\wedge dy_1\wedge dy_2\wedge dy_3\wedge dy_4\wedge\epsilon_2\,. \tag{4.16}$$

The near-horizon region of this solution is obtained by keeping the following quantities fixed as $\alpha'\to 0$:

$$g\,,\quad \tilde{x}^\mu = \frac{x^\mu}{\sqrt{\alpha'}}\,,\quad \tilde{y}^k = \frac{y^k}{\sqrt{\alpha'}}\,,\quad \tilde{r} = \frac{r}{\sqrt{\alpha'}}\,, \tag{4.17}$$

and take the critical scaling limit

$$\frac{\theta}{\alpha'}\to 1\,, \tag{4.18}$$

such that

$$h = 1-(H_1H_5)^{-1}\,. \tag{4.19}$$

The metric (4.16) interpolates between $AdS_3\times S^3\times T^4$ in the IR ($\tilde{r}\sim 0$) and an array of F1-strings in the UV ($\tilde{r}\sim\infty$), which are stretched in the x^1 direction and wrapped on T^4. In the UV limit $\tilde{r}\to\infty$ we recognize (2.11) by setting $\epsilon = h$ and $\lambda = 1$. From this solution we calculate the open string data in (2.8) and (2.9):

$$\frac{ds^2(\tilde{G})}{\alpha'} = (H_1H_5)^{-1/2}\eta_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu\,, \tag{4.20}$$
$$\tilde{G}_{\mathrm{O}} = g\sqrt{\frac{H_1}{H_5}}\,,\qquad \tilde{\Theta} = \epsilon^{\mu\nu}\partial_\mu\partial'_\nu\,. \tag{4.21}$$

The open string metric and coupling indeed approaches constant values when $\tilde{r}\to\infty$, as expected for a NCOS theory, and the noncommutativity parameter remains fixed along the flow to the IR. We also note that by introducing a fundamental (fixed) constant with unit length, and using this to define fixed canonical coordinates it follows from (4.20) and (4.21), that the Regge slope and noncommutativity parameter of the NCOS are equal.

5. Deformation of near-horizon geometries

In many situations one only has access to the brane solutions in the (undeformed) near-horizon region, such as for example the various AdS_5 vacua of type-IIB supergravity. Since these types of solutions are dual to field theories, and since field theories do not admit spatio-temporal noncommutativity deformations [44], the deformation procedure described above is only directly applicable in the magnetic case, whereas an electric deformation will lead to a singularity at a finite scale. This may be understood from the form of the line element in the brane directions (3.14), that is unbounded from above in the near-horizon region, i.e. $0 < \mathcal{E} < \infty$. Hence, for any electric deformation with finite parameter θ/α' — no matter how small — there will be a critical scale $E_{\rm crit} = \alpha'/\theta$ where the deformed configuration has an essential singularity (note that in fixed units the critical scale is $u_{\rm crit} = 1/\sqrt{\theta}$).

Interestingly, an electric deformation of the $AdS_5 \times S^5$ with parameter θ/α' is S-dual to the near-horizon limit of a magnetically deformed D3-brane metric with harmonic function with negative integration constant $-(\theta/\alpha')^2$. The metrics yield (2.10) and (2.11), but are pathological in other ways: the asymptotic electric geometry is singular instead of the smeared string (3.23) and the magnetic probe brane has infinite negative vacuum energy in the UV so it is unstable.

In the case of D3-branes, we now wish to find some criteria for when the strong coupling limit of a noncommutative field theory on the brane has a weak coupling dual which is a noncommutative open string theory with an electric supergravity dual that is related by type-IIB S-duality to the magnetic supergravity dual. This is true in the maximally supersymmetric case (provided the axion is rational [43]). As we shall show next, under certain conditions the same is true also for extremal D3-branes with less symmetry. In order to examine this in detail, we start from the general D3-brane configuration in (3.2) and assume (3.15). We make one further main assumption, namely that the axion is vanishing (which actually could be relaxed to a rational axion [43]):

$$C_0 = 0 \,. \tag{5.1}$$

We then find (see appendix B for details) that the magnetic deformation of the brane and the electric deformation of its S-dual, are S-dual (modulo a diffeomorphism) provided that the undeformed dilaton is constant and the zero-force condition is obeyed:

$$e^{\phi} = g = \text{constant}\,, \qquad V = g^{-1}\sqrt{-\det g_{\mu\nu}} - \omega = 0\,. \tag{5.2}$$

We also learn that the magnetic deformation parameter θ and the electric parameter $\tilde{\theta}$ are related as follows:

$$\tilde{\theta} = \frac{\alpha' g}{\sqrt{1 + (\alpha'/\theta)^2}} \equiv \sin\nu \ \alpha' g\,. \tag{5.3}$$

We also find certain conditions on the metric and the transverse potentials in the undeformed configuration (see eqs. (B.5), (B.11) and (B.14) in appendix B), and that the coordinates $\tilde{x}^\mu$ in the electric supergravity dual and x^μ in the magnetic dual must be related as follows:

$$\tilde{x}^\mu = \sqrt{\cos\nu}\, x^\mu\,, \qquad \tilde{r} = \frac{r}{\sqrt{\cos\nu}}\,. \tag{5.4}$$

All this implies that the magnetic near-horizon limit (3.16) is mapped under S-duality to the electric near-horizon limit (3.22). We remark that (3.22) only requires $\tilde{\theta}/\alpha' \to 1$, whereas the precise critical scaling in (5.3) is required by S-duality. Hence for a D3-brane configuration obeying (5.2), (B.5), (B.11) and (B.14), the magnetic and electric near horizon geometries are S-dual, and by exploiting this fact in the extreme UV limit it follows that the corresponding NCYM and NCOS theories must therefore be S-dual. We can summarize the above results in the following commutative diagram:

$$
\begin{tikzcd}
& & \mathrm{AdS}_5 \times M^5 \arrow[drr, leftrightarrow, "\Lambda_{\mathrm{m}}"] & & \\
i^*\mathrm{D3} \arrow[urr, "\ell im"] \arrow[rr, leftrightarrow, "\Lambda_{\mathrm{m}}"] \arrow[d, leftrightarrow, "\mathrm{S}"'] & & i^*\Lambda_{\mathrm{m}}(\mathrm{D3}) \arrow[rr, "\ell im_{\mathrm{m}}"] \arrow[d, leftrightarrow, "\mathrm{S}"'] & & \begin{array}{c}\text{NCYM}\\ \text{supergravity}\\ \text{dual}\end{array} \arrow[d, leftrightarrow, "\mathrm{S}"'] \\
i^*S(\mathrm{D3}) \arrow[rr, leftrightarrow, "\Lambda_{\mathrm{e}}"'] & & i^*\Lambda_{\mathrm{e}}(S(\mathrm{D3})) \arrow[rr, "\ell im_{\mathrm{e}}"'] & & \begin{array}{c}\text{NCOS}\\ \text{supergravity}\\ \text{dual}\end{array}
\end{tikzcd}
$$

Here Λ_{m} and Λ_{e} denote the magnetic and electric deformations, and ℓim_{m} and ℓim_{e} the magnetic and electric near horizon limits (3.16) and (3.22), respectively. The field theory near-horizon limit ℓim is identical to the magnetic near-horizon limit only for $\theta = 0$. The operation i^*, where i is defined by (3.21), indicates that we are only considering the brane configuration at the vanishing locus of the brane potential in (5.2). Importantly, from the discussion in section 3.3 (see eq. (3.30)) it follows that i^* commutes with the deformations Λ_{m} and Λ_{e}, so that this restriction is indeed well-defined, in the sense that the NCYM and NCOS supergravity duals have well-defined UV limits.

As already mentioned, the simplest non-trivial example of the above S-duality is the maximally symmetric extremal D3-brane. We shall next turn to a non-trivial example involving extremal branes with less symmetry and a non-trivial probe brane potential.

6. Open strings in a $\mathcal{N} = 1$ background

In this section we examine noncommutative deformations of the Pilch-Warner (PW) solution [7]. The PW solution has conformal $\mathcal{N} = 1$ supersymmetry and is a warped solution with a four-dimensional vanishing locus in the internal space.

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6.1 The PW solution

We use the conventions of [45], except that we use $(- + \cdots +)$ signature. The undeformed supergravity dual of the $\mathcal{N} = 1$ SCFT [7] in the string frame is given by:

$$\begin{aligned}
ds^2 &= \alpha'\Omega^2 \left[\frac{u^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{u^2}du^2 + R^2 d\hat{s}_5^2 \right], \\
e^{2\phi} &= g^2\,, \quad B_2 = \alpha'\beta_2\,, \quad C_2 = \alpha'\gamma_2\,, \\
C_4 &= (\alpha')^2 \left[\frac{k}{g}\frac{u^4}{R^4} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + \gamma_4 \right], \\
C_6 &= (\alpha')^3 \left[\alpha_2 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \right],
\end{aligned} \tag{6.1}$$

where ds_5^2 is the metric on a deformed five-sphere and Ω^2 is a 'warp-factor' depending on one of the sphere coordinates:

$$\begin{aligned}
d\hat{s}_5^2 &= \frac{2}{3}\Bigg\{ \left(d\alpha^2 + \frac{\cos^2\alpha}{3-\cos(2\alpha)}((\sigma^1)^2 + (\sigma^2)^2) + \frac{\sin^2(2\alpha)}{(3-\cos(2\alpha))^2}(\sigma^3)^2 \right) + \\
&\quad + \frac{2}{3}\left(d\phi + \frac{2\cos^2\alpha}{3-\cos(2\alpha)}\sigma^3 \right)^2 \Bigg\}, \qquad (6.2) \\
\Omega^2 &= 2^{1/3}\sqrt{1 - \frac{1}{3}\cos(2\alpha)}\,. \qquad (6.3)
\end{aligned}$$

Here $0 \leq \alpha \leq \pi/2$ and σ^i, $i = 1, 2, 3$ are the SU(2) invariant forms satisfying $d\sigma^1 = \sigma^2 \wedge \sigma^3$. The constants[10] are

$$R = 2^{-5/3}3R_0\,, \qquad R_0^4 = 4\pi g_s N\,, \qquad k = 2^{5/3}3^{-1}\,, \tag{6.4}$$

where R_0 is the radius of the round five-sphere in the $\mathcal{N} = 8$ vacuum. Note that all quantities have been given in units of α' such that they are fixed in the near-horizon region. Let us therefore define

$$U = \frac{\Omega u}{R}\,, \qquad \omega = \frac{ku^4}{gR^4}\,. \tag{6.5}$$

The zero-force condition (5.2) implies

$$V = \frac{u^4}{gR^4}(\Omega^4 - k) = 0 \;\Rightarrow \alpha = 0\,. \tag{6.6}$$

From (6.2) it follows that the vanishing locus is a codimension 2 manifold in the transverse space, with frame du, $\sigma^{1,2}$ and $d\phi + \sigma^3$.

[10] There is a typographical error in the expression for the five-form normalization constant m given in [7]; the correct value is $m = 2^{10/3}3^{-2}R_0^{-1}$ which corresponds to the value of k given below.

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6.2 Deforming the PW solution

We now carry out the transformation procedure described in section 3 to determine the magnetically deformed near-horizon geometry

$$
\begin{aligned}
ds'^2 &= \alpha'\Omega^2\left[\frac{u^2}{R^2}\left(-(dx^0)^2+(dx^1)^2+\frac{1}{h}((dx^2)^2+(dx^3)^2)\right)+\frac{R^2}{u^2}du^2+R^2d\hat{s}_5^2\right],\\
e^{2\phi'} &= \frac{g^2}{h},\\
B_2' &= \alpha'\left[-\frac{\theta U^4}{h}dx^2\wedge dx^3+\beta_2\right], \quad C_2'=\alpha'\left[-\theta\omega dx^0\wedge dx^1+\gamma_2\right],\\
C_4' &= (\alpha')^2\left[\frac{\omega}{h}dx^0\wedge dx^1\wedge dx^2\wedge dx^3+\gamma_4+\frac{\theta U^4}{h}\gamma_2\wedge dx^2\wedge dx^3-\theta\alpha_2\wedge dx^0\wedge dx^1\right],
\end{aligned}
\tag{6.7}
$$

where the deformation parameter θ has dimension (length)^2 and we have defined

$$h = 1+\theta^2U^4\,. \tag{6.8}$$

The solution (6.7) on the vanishing locus (6.6), is interpreted as the supergravity dual of an $\mathcal{N}=1$ noncommutative Yang-Mills theory.

As discussed in section 5, we can now obtain the supergravity dual of an $\mathcal{N}=1$ electrically deformed theory by S-dualizing the solution (6.7). The S-duality rules are given by:

$$
\begin{aligned}
d\tilde{s}^2 &= e^{-\phi'}ds'^2\,,\\
e^{\tilde{\phi}} &= \frac{e^{\phi'}}{C_0'^2+e^{2\phi'}}\,, \quad \tilde{C}_0=-\frac{C_0'}{C_0'^2+e^{2\phi'}}\,,\\
\tilde{B}_2 &= C_2'\,, \qquad \tilde{C}_2=-B_2'\,,\\
\tilde{C}_4 &= C_4'+B_2'\wedge C_2'\,.
\end{aligned}
\tag{6.9}
$$

We find the following electric near-horizon geometry:

$$
\begin{aligned}
d\tilde{s}^2 &= \alpha'\left[\tilde{\mathcal{E}}^2\left(\frac{-(d\tilde{x}^0)^2+(d\tilde{x}^1)^2}{\tilde{h}}+(d\tilde{x}^2)^2+(d\tilde{x}^3)^2)\right)+\frac{\Omega^4}{k}\tilde{\mathcal{E}}^{-2}(d\tilde{r}^2+\tilde{r}^2d\hat{s}_5^2)\right],\\
e^{2\tilde{\phi}} &= \frac{\tilde{g}^2}{\tilde{h}},\\
\tilde{B}_2 &= \alpha'\left[-\frac{k}{\Omega^4}\frac{\tilde{\mathcal{E}}^4}{\tilde{h}}d\tilde{x}^0\wedge d\tilde{x}^1+\gamma_2\right],\\
\tilde{C}_2 &= \alpha'\left[\frac{\Omega^4}{k}\tilde{\omega}d\tilde{x}^2\wedge d\tilde{x}^3-\beta_2\right],\\
\tilde{C}_4 &= (\alpha')^2\left[\frac{\tilde{\omega}}{\tilde{h}}d\tilde{x}^0\wedge d\tilde{x}^1\wedge d\tilde{x}^2\wedge d\tilde{x}^3+\gamma_4+\beta_2\wedge\gamma_2-\frac{k}{\Omega^4}\frac{\tilde{\mathcal{E}}^4}{\tilde{h}}d\tilde{x}^0\wedge d\tilde{x}^1\wedge(\alpha_2+\beta_2)\right],
\end{aligned}
\tag{6.10}
$$

where we have performed the reparametrization

$$\tilde{x}^\mu = \theta^{-1/2} x^\mu \,, \qquad \tilde{r} = k^{1/2}\theta^{1/2} u \,, \tag{6.11}$$

and defined

$$\begin{aligned} \tilde{g} &= g^{-1} \,, \qquad \tilde{R} = g^{-1/2} R \,, \\ \tilde{\mathcal{E}}^2 &= \left(1 + \frac{k^2 \tilde{R}^4}{\Omega^4 \tilde{r}^4}\right)^{-1/2} \,, \qquad \tilde{\omega} = \frac{k}{\Omega^4} \frac{\tilde{\mathcal{E}}^4}{\tilde{g}} \,, \qquad \tilde{h} = 1 - \tilde{\mathcal{E}}^4 \,. \end{aligned} \tag{6.12}$$

This is interpreted as providing the $\mathcal{N} = 1$ supergravity dual on the vanishing locus (6.6), from which one might extract a noncommutative open string theory with $\mathcal{N} = 1$ supersymmetry. To investigate this and illustrate the importance of the zero-force condition, we proceed by computing open string quantities for a probe brane in this background. This yields the following open string metric ($\alpha, \beta = 0, 1$ and $a, b = 2, 3$):

$$\frac{\tilde{G}_{\mu\nu}}{\alpha'} = \tilde{\mathcal{E}}^2 \left[\left(1 + \frac{\tilde{\mathcal{E}}^4}{\tilde{h}}(1 - \frac{k^2}{\Omega^8})\right) \eta_{\alpha\beta} \oplus \delta_{ab} \right] , \tag{6.13}$$

open string coupling:

$$\tilde{G}_{\mathrm{O}} = \tilde{g} \sqrt{\left| 1 + \frac{\tilde{\mathcal{E}}^4}{\tilde{h}}(1 - \frac{k^2}{\Omega^8}) \right|} \,, \tag{6.14}$$

and the spatio-temporal noncommutativity parameter

$$\tilde{\Theta}^{\alpha\beta} = \epsilon^{\alpha\beta} \frac{k}{\Omega^4} \left(1 + \frac{\tilde{\mathcal{E}}^4}{\tilde{h}}(1 - \frac{k^2}{\Omega^8})\right)^{-1} . \tag{6.15}$$

Note that (6.13) and (6.15) are dimensionless in the dimensionless $\tilde{x}^\mu$ coordinates. To have a noncommutative open string theory one requires that these quantities are finite in the limit $\tilde{h} \to 0$. This can only occur when

$$\Omega^4 = k \,, \tag{6.16}$$

i.e. when the zero-force condition (6.6) is satisfied. When this equation is obeyed, the open string data becomes:

$$\frac{\tilde{G}_{\mu\nu}}{\alpha'} = \tilde{\mathcal{E}}^2 \eta_{\mu\nu} \,, \qquad \tilde{G}_{\mathrm{O}} = \tilde{g} \,, \qquad \tilde{\Theta}^{\alpha\beta} = \epsilon^{\alpha\beta} \,, \tag{6.17}$$

which is exactly what we should expect recalling the discussion in subsection 3.2. Note that, if one starts from a full brane solution, as in e.g. [40], we expect to find (6.17) in all directions, not only on the locus. The reason we get (6.13) to (6.15) is that we here perform the S-duality transformation also off the locus where it is actually not valid, see section 5 and appendix B.

7. Discussion

In this paper we have described how the limits on a D-brane required for a noncommutative theory naturally arise when considering a probe brane in an appropriate background supergravity solution. We describe how to construct such solutions and in particular we use this construction to find deformed $\mathcal{N}=1$ supersymmetric solutions. It should be stressed that the decoupling of the probe brane from the bulk only occurs when it is in the near horizon region, with the NC theories appearing in the UV limit of this region. Nevertheless, one may in the near horizon region investigate the D-brane in terms of the open strings ending on it, and the corresponding *open string data*, at any radial location. The noncommutative deformation is shown to be independent of the probe brane radial distance. When one considers electrically deformed solutions one sees that the noncommutative open string limit only occurs on D-branes that are embedded in such a way that they have vanishing potential energy and so obey the zero force condition. This is shown also for the (conformal) $\mathcal{N}=1$ supersymmetric theory. In field theory, the interpretation would be that the ultraviolet completion of the theory to a noncommutative open string theory only holds for a submanifold in moduli space where the potential energy vanishes. It should also be noted that the primary example we have investigated, the Pilch-Warner solution, is the dual of a very specific $\mathcal{N}=1$ theory that can be viewed as a relevant deformation from the $\mathcal{N}=4$ theory. It would be interesting to look at other examples of $\mathcal{N}=1$ theories and the results of their deformations.

In the context of $\mathcal{N}=1$ theories, for which the full brane solution is not known, only the magnetically deformed theory can be obtained by means of the T-duality transformation applied to the near horizon solution. Hence one has to rely on other methods to find the electrically deformed solution and in this paper this is done by applying an S-duality transformation to the magnetic solution. However, as explained in section 5 this works only on the vanishing locus of the field theory potential, and it therefore also follows that the NCYM and NCOS theories are S-dual in the UV limit only on the locus of vanishing potential.[11]

Throughout this paper we have always implicitly been in the large N limit. An interesting further area of study would be to extend the above analysis to include $1/N$ corrections and see the role they have in the existence of the open string limit and in the independence of the deformation on the radial direction. The construction of the deformed solution corresponding to $1/N$ corrections would proceed as above beginning from an undeformed solution to supergravity with α' corrections.

Another aspect not explored in this paper is the possibility of an $\mathrm{SL}(2,\mathbb{Z})$-covariant analysis. For example, if one considers describing the D3-brane with

[11] Note that although the zero-force condition is not satisfied off the locus, the open string quantities defined in section 2 nevertheless seem to behave the same way in all directions as follows from e.g. [40].

open D-strings then one sees how the noncommutative 'magnetic' theory has a description in terms of noncommutative open D-string theory at strong coupling. An $SL(2,\mathbb{Z})$-invariant description would require describing the brane fluctuations with an $SL(2,\mathbb{Z})$-inert object such as an open D3-brane. This is work in progress.

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A. The noncommutative scalar fields

In order to derive the field redefinition in the scalar sector on a noncommutative Dp-brane we start from a 9-brane in a constant background described by the Born-Infeld lagrangian

$$S_9 = -\mu_9 \int_{M^{10}} d^{10}x e^{-\check{\phi}} \sqrt{-\det\left[\frac{\check{g}_{MN}}{\alpha'} + \frac{\check{B}_{MN}}{\alpha'} + F_{MN}\right]}, \tag{A.1}$$

where M^{10} is a ten-dimensional space-time with metric $\check{g}_{MN}$, dilaton $\check{\phi}$, two-form potential $\check{B}_{MN}$ and field strength $F_{MN} = \partial_M A_N - \partial_N A_M$. The $(p+1)$-dimensional Born-Infeld lagrangian on $M^{p+1} \times \mathbf{R}^{9-p}$ with non-vanishing two-form fluxes only on M^{p+1} is obtained by double dimensional reduction on $M^{10} = M^{p+1} \times T^{9-p}$ followed by decompactifying the resulting T-dual torus keeping the effective Dp-brane tension μ_p fixed. Thus, if we denote the indices on M^{p+1} by $\mu = 0, \ldots, p$ and the indices on T^{9-p} by $i = p+1, \ldots, 9$, and take

$$\check{g}_{\mu i} = 0\,, \quad \check{B}_{\mu i} = \check{B}_{ij} = 0\,, \tag{A.2}$$

then we can identify the metrics and the two-form potentials on the M^{p+1},

$$g_{\mu\nu} = \check{g}_{\mu\nu}\,, \qquad B_{\mu\nu} = \check{B}_{\mu\nu}\,, \tag{A.3}$$

and relate the metric g_{ij} and the scalar coordinates Φ^i (with the dimension of energy) on the T-dual torus to the ten-dimensional cometric and internal vector field components on T^{9-p} as follows:

$$g_{ij} = (\alpha')^2 \check{g}^{ij}\,, \quad \Phi^i = A_i\,. \tag{A.4}$$

As a result

$$S_9 \overset{T^{9-p}}{\longrightarrow} S_p = -\mu_p \int_{M^{p+1}} d^{p+1}x e^{-\phi} \sqrt{-\det\left[\frac{g_{\mu\nu}}{\alpha'} + \partial_\mu \Phi^i \partial_\nu \Phi^j \frac{g_{ij}}{\alpha'} + \frac{B_{\mu\nu}}{\alpha'} + F_{\mu\nu}\right]}\,, \tag{A.5}$$

where we have identified the T-dual dilaton and the Dp-brane tension as follows:

$$\mu_p e^{-\phi} = \mu_9 \mathrm{Vol}_{9-p}(\check{g}) e^{-\check{\phi}}\,, \qquad \mathrm{Vol}_{9-p}(\check{g}) = \int_{T^{9-p}} d^{9-p}x \sqrt{\det \frac{\check{g}_{ij}}{\alpha'}}\,. \tag{A.6}$$

We next recall [29] that if one takes the B-fluxes to be non-zero in the spatial directions $m = p-1, p$ (so we assume $p > 1$) and considers the following decoupling limit on the original 9-brane:

$$\ell im_9 \ : \quad \frac{\check{g}_{MN}}{\alpha'} \sim \begin{cases} \epsilon^{-1/2} & \text{for } M, N \neq p-1, p \\ \epsilon^{1/2} & \text{else} \end{cases}\,, \qquad \frac{\check{B}_{mn}}{\alpha'} \sim \epsilon^0\,, \qquad e^{\check{\phi}} \sim \epsilon^{-1}\,, \tag{A.7}$$

then (A.1) reduces (up to total derivatives and higher derivative terms) to the non-commutative Maxwell action in ten dimensions:

$$S_9 \overset{\ell im_9}{\longrightarrow} \hat{S}_9 = -\frac{1}{4} \int_{M^{10}} d^{10}x \frac{1}{\check{G}_{\mathrm{O}}} \sqrt{-\det \frac{\check{G}}{\alpha'}} (\alpha')^2 \check{G}^{MN} \check{G}^{PQ} \hat{F}_{MP} \hat{F}_{NQ}\,, \tag{A.8}$$

where $\check{G}$ and $\check{G}_{\mathrm{O}}$ are the ten-dimensional open string metric and coupling, and the noncommutative ten-dimensional field strength is

$$\hat{F}_{MN} = \partial_M \hat{A}_N - \partial_N \hat{A}_M + \hat{A}_M \star \hat{A}_N - \hat{A}_N \star \hat{A}_M\,. \tag{A.9}$$

Here the noncommutative vector potential $\hat{A}_M$ is related to A_M by the non-local field redefinition:

$$\hat{A}_M = A_M - \frac{1}{2}\theta^{NP} A_N (2\partial_P A_M - \partial_M A_P) + \mathcal{O}(\theta^2)\,, \tag{A.10}$$

where θ^{MN} is the ten-dimensional Poisson structure. From eqs. (A.2)-(A.4) and (A.6) it follows that the ten-dimensional open string data are related to the T-dual $(p+1)$-dimensional open string data as follows:

$$\check{G}_{MN} = G_{\mu\nu} \oplus \check{g}_{ij}\,, \tag{A.11}$$

$$\check{G}_{\mathrm{O}} = \mathrm{Vol}_{9-p}(\check{G}) G_{\mathrm{O}} = \mathrm{Vol}_{9-p}(\check{g}) G_{\mathrm{O}}\,, \tag{A.12}$$

$$\theta^{MN} = \theta^{\mu\nu} \oplus 0\,, \qquad \theta^{\mu\nu} = \begin{cases} \theta^{mn} & \text{for } \mu, \nu = m, n \\ 0 & \text{else} \end{cases}\,. \tag{A.13}$$

Moreover, from (A.3), (A.4) and (A.6) it follows that (A.7) is equivalent to the following $(p+1)$-dimensional decoupling limit:

$$\ell im_p \ : \qquad g_{\mu\nu}/\alpha' \sim \begin{cases} \epsilon^{-\frac{1}{2}} & \text{for } \mu, \nu = 0, \ldots, p-2 \\ \epsilon^{\frac{1}{2}} & \text{for } \mu, \nu = p-1, p \end{cases}\,, \tag{A.14}$$

$$g_{ij}/\alpha' \sim \epsilon^{\frac{1}{2}}\,, \quad B_{mn}/\alpha' \sim \epsilon^0\,, \quad e^{\phi} \sim \epsilon^{\frac{1}{4}(5-p)}\,,$$

which we identify as the NCYM limit (2.10) on a Dp-brane with B-flux of rank two such that (again up to total derivatives and higher derivative terms)

$$S_p \xrightarrow{\ell im_p} \hat{S}_p \,, \tag{A.15}$$

where $\hat{S}_p$ defines the noncommutative Yang-Mills theory on the Dp-brane. We have thus established the 'commutative' diagram:

$$\begin{tikzcd} S_9 \arrow[r, "\ell im_9"] \arrow[d, "T^{9-p}"'] & \hat{S}_9 \arrow[d, "T^{9-p}"'] \\ S_p \arrow[r, "\ell im_p"] & \hat{S}_p \end{tikzcd} \tag{A.16}$$

Hence, we may obtain explicitly the form of $\hat{S}_p$ and the field redefinition of the noncommutative scalar fields on the Dp-brane, by double dimensional reduction of (A.8)–(A.10) using (A.11)–(A.13), that is

$$\hat{S}_p = -\int_{M^{p+1}} d^{p+1}x \frac{1}{g^2_{\mathrm{YM}}} \sqrt{-\det \bar{G}} \left(\frac{1}{4} \bar{G}^{\mu\nu} \bar{G}^{\rho\sigma} \hat{F}_{\mu\rho} \hat{F}_{\nu\sigma} + \frac{1}{2} \bar{G}^{\mu\nu} \hat{D}_\mu \hat{\Phi}^i \hat{D}_\nu \hat{\Phi}^j \bar{g}_{ij} \right) , \tag{A.17}$$

$$\hat{\Phi}^i = \Phi^i - \theta^{mn} A_m \partial_n \Phi^i + \mathcal{O}(\theta^2) \,, \tag{A.18}$$

$$\hat{D}_\mu \hat{\Phi}^i = \partial_\mu \hat{\Phi}^i + \hat{A}_\mu \star \hat{\Phi}^i - \hat{\Phi}^i \star \hat{A}_\mu \,,$$

and $\bar{G}_{\mu\nu}$ and g^2_{YM} are defined as in (4.10) and (4.12) and $\bar{g}_{ij}$ as in (3.18). To be precise, we have the following scaling behaviour:

$$\frac{G_{\mu\nu}}{\alpha'} \sim \epsilon^{-\frac{1}{2}} \ell^{-2} \eta_{\mu\nu} \,, \qquad \frac{g_{ij}}{\alpha'} \sim \epsilon^{\frac{1}{2}} \ell^2 \bar{g}_{ij} \,, \qquad G_{\mathrm{O}} \sim \epsilon^{-\frac{p-3}{4}} \,, \tag{A.19}$$

where ℓ is a fixed length scale such that $g^2_{\mathrm{YM}} \sim \ell^{p-3}$.

The condition of constant background is not crucial. The important input is the nature of the limit (A.14), and the fact that the θ parameter is constant. Let us start from a Dp-brane supergravity dual with UV limit (A.14), and expand the Born-Infeld lagrangian in this limit. From the results in section 3 we find

$$S_p = -\int d^{p+1}\xi e^{-\phi} \sqrt{-\det \left(\frac{g_{\mu\nu}}{\alpha'} + \frac{B_{\mu\nu}}{\alpha'} + S_{\mu\nu} + F_{\mu\nu} + \beta_{\mu\nu} \right)}$$

$$= -\int d^{p+1}\xi e^{-\phi}\sqrt{-\det\left(\frac{g_{\mu\nu}}{\alpha'}+\frac{B_{\mu\nu}}{\alpha'}\right)}\sqrt{\det\left(1+\left(\frac{g}{\alpha'}+\frac{B}{\alpha'}\right)^{-1}(S+F+\beta)\right)}$$

$$= -\int d^{p+1}\xi \frac{1}{G_O}\sqrt{-\det\frac{G}{\alpha'}}\Big(\sqrt{\det(1+\theta(F+\beta))}+(\alpha')^2 G^{\mu\nu}G^{\rho\sigma}\hat{F}_{\mu\rho}\hat{F}_{\nu\sigma}+ + (\alpha')^2 G^{\mu\nu}\hat{D}_\mu\hat{\Phi}^i\hat{D}_\nu\hat{\Phi}^j\bar{g}_{ij}+\cdots\Big)\,, \tag{A.20}$$

where

$$S_{\mu\nu}=\alpha'\partial_\mu\Phi^i\partial_\nu\Phi^j g_{ij}\,,\qquad \beta_{\mu\nu}=\alpha'\partial_\mu\Phi^i\partial_\nu\Phi^j B_{ij}\,. \tag{A.21}$$

In expanding the determinant we have used the fact that θ and F are fixed in the UV while S, β and $\alpha' G^{-1}$ are subleading. The leading order, which is divergent, consists of traces of the form $\mathrm{tr}((\theta(F+\beta))^n)$, where we keep β. These terms give the divergent $\sqrt{\det(1+\theta(F+\beta))}$-term in (A.20). In the first subleading order we find vanishing traces of the form $\mathrm{tr}(\theta S(\theta(F+\beta))^n)$ and $\mathrm{tr}(\alpha' G^{-1}(F+\beta)(\theta(F+\beta))^n)$, where n is an integer and all possible orderings occur. The next order, which is finite, consists of traces of the form

$$\mathrm{tr}(G^{-2}F^2(\theta F)^n)\,,\quad \mathrm{tr}(G^{-1}S(\theta F)^n)\ \text{and}\quad \mathrm{tr}(S^2\theta^2(\theta F)^n)\,, \tag{A.22}$$

multiplied by traces $\mathrm{tr}(\theta F)^m$, where m, n are integers. Here we can drop β, since the scaling of the structures in (A.22) is precisely cancelled by the determinant prefactor in (A.20), so that the β contributions vanish in the UV. These terms give the finite kinetic terms in (A.20), modulo derivatives of G^{-1}, g_{ij} and ϕ with respect to internal coordinates Φ^i. The reason is that in forming the kinetic terms one needs to rewrite the structures in (A.22) by moving derivatives from F to the scalar fields by integrating by parts. Note, however, that in doing so, one need not bother about differentiating with respect to u, since that lowers the degree of divergence (recall that $\partial_\mu u$ is fixed). Hence, in the case of maximally symmetric extremal branes discussed in section 4.1, this implies that the UV limit gives the same result (A.17) as for a flat background. In cases with less symmetry, as for instance the $T^{(1,1)}$ and the Pilch-Warner solutions a more careful analysis is required to obtain the full kinetic term for the noncommutative scalars.

B. Conditions for S-duality

We are interested in finding some conditions for which the S-dual of a magnetically deformed solution is equal to the electric deformation of the S-dual of the original solution modulo a *brane isomorphism*. By a brane isomorphism we mean a diffeomorphism that preserves the structure of the brane solution. This may be written as follows:

$$S(\Lambda_{\mathrm{m}}(\mathrm{D3}))=\varphi^*(\Lambda_{\mathrm{e}}(S(\mathrm{D3})))\,, \tag{B.1}$$

where D3 denotes a (not necessarily self-dual) 3-brane configuration as in (3.2) (note that $\gamma_6 = 0$ for a D3-brane), $\Lambda_{\mathrm{m,e}}$ are magnetic and electric $\mathrm{O}(p+1, p+1)$ transformations and φ is a brane isomorphism, which by definition acts as a diffeomorphism on the transverse space and a linear transformation on the brane world volume as follows:

$$\varphi^* dx^\mu = M^\mu{}_\nu dx^\nu \,. \tag{B.2}$$

In order to simplify the analysis we assume that the axion vanishes:

$$C_0 = 0 \,, \qquad \gamma_8 = 0 \,. \tag{B.3}$$

For clarity, we also include the NS six-form potential, which for a D3-brane must have the form:

$$B_6 = dx^0 \wedge \cdots \wedge dx^3 \wedge \delta_2 \,. \tag{B.4}$$

Using the $\mathrm{O}(p+1, p+1)$ transformation rules (3.11) and (3.13) and the S-duality transformation rules (6.9) we find from (B.1) that φ has to be a symmetry of the transverse potentials:

$$\varphi^*(\beta_2) = \beta_2 \,, \qquad \varphi^*(\gamma) = \gamma \,. \tag{B.5}$$

By examining the conditions on the brane part of the metric and the dilaton in the asymptotically flat region we then determine

$$M^\mu{}_\nu = \sqrt{\cos\nu}\,\delta^\mu_\nu \,, \qquad \cos^2\nu = \frac{1}{1+(\theta/\alpha')^2} \,, \tag{B.6}$$

$$\tilde\theta = \pm \cos\nu \; \theta g \,, \tag{B.7}$$

where θ is the magnetic parameter, $\tilde\theta$ the electric parameter and g the asymptotic value of the dilaton in the undeformed D3-brane configuration. The conditions on the metric in the electric and magnetic brane directions and the dilaton equation then reads:

$$\begin{aligned} e^{-\phi} h^{\frac{1}{2}} \mathcal{E}^2 &= e^{-\tilde\phi} \frac{\tilde{\mathcal{E}}^2}{\tilde h} \cos\nu \,, \qquad e^{-\phi} h^{-\frac{1}{2}} \mathcal{E}^2 = e^{-\tilde\phi} \tilde{\mathcal{E}}^2 \cos\nu \,, \\ h\tilde h &= e^{2(\phi-\tilde\phi)} \,, \end{aligned} \tag{B.8}$$

where

$$\begin{aligned} &\tilde{\mathcal{E}} \equiv \varphi^* \mathcal{E} \,, \qquad \tilde\phi = \varphi^* \phi \,, \\ &h = 1 + \left(\frac{\theta}{\alpha'}\right)^2 \mathcal{E}^4 \,, \quad \tilde h = 1 - \left(\frac{\tilde\theta}{\alpha'}\right)^2 e^{-2\tilde\phi} \tilde{\mathcal{E}}^4 \,. \end{aligned} \tag{B.9}$$

These equations are equivalent to

$$h\tilde h = 1 \,, \quad \phi = \tilde\phi \,, \tag{B.10}$$

$$\tilde{\mathcal{E}}^4 = \frac{1}{1+\cos^2\nu(\mathcal{E}^{-4}-1)} \,. \tag{B.11}$$

We now turn to the remaining conditions on the potentials. From the electric and magnetic directions of the two-form potentials and the brane part of the four-form potential we get the conditions

$$
\begin{aligned}
-\theta\omega &= \tilde{\theta}\frac{e^{-2\tilde{\phi}}\tilde{\mathcal{E}}^4}{\tilde{h}}\cos\nu\ , \quad \theta\frac{\mathcal{E}^4}{h} = -\tilde{\theta}\tilde{\omega}\cos\nu\ , \\
\omega &= \frac{\tilde{\omega}}{\tilde{h}}\cos^2\nu\ ,
\end{aligned} \tag{B.12}
$$

where $\tilde{\omega} = \varphi^*\omega$. Using (B.10) and (B.11) these equations are equivalent to (5.2). In the six-form sector we find the condition

$$
-\theta\omega\alpha_2 = \tilde{\theta}\varphi^*\delta_2\ , \tag{B.13}
$$

which is identically satisfied when the lower rank form equations are obeyed. Finally, the metric equation in the transverse directions amounts to

$$
\varphi^* i^*(\mathcal{E}^2 ds_6^2) = \frac{1}{\cos\nu} i^*(\mathcal{E}^2 ds_6^2))\ . \tag{B.14}
$$

The condition (B.14) can be solved by taking

$$
i^* ds_6 = (i^*\mathcal{E})^{-2} d\bar{s}^2_{6-n}\ , \qquad d\bar{s}^2_{6-n} = dr^2 + r^2 d\bar{s}^2_{5-n}\ , \tag{B.15}
$$

where n is the codimension of the vanishing locus; r is a radial coordinate acted on by the brane isomorphism by the scale transformation

$$
\varphi^* r = \frac{1}{\sqrt{\cos\nu}} r\ ; \tag{B.16}
$$

and $d\bar{s}^2_{5-n}$ is an invariant metric on the remaining $5-n$ dimensions in the locus. Then (B.11) implies that the vierbein $\mathcal{E}$ is 'harmonic' on the locus:

$$
i^*\mathcal{E}^2 = (1 + \frac{k(\alpha')^2}{r^4})^{-\frac{1}{2}}\ . \tag{B.17}
$$

In the magnetic near-horizon limit (3.16) we can thus identify

$$
u = \frac{\mathcal{E}}{\sqrt{\alpha'}} \sim \frac{r}{\sqrt{k\alpha'}}\ . \tag{B.18}
$$

From (B.6), which is equivalent to

$$
\varphi^* dx^\mu = \sqrt{\cos\nu}\ dx^\mu\ , \tag{B.19}
$$

and (B.7) and (B.16) it follows that the S-dual of the magnetic near-horizon limit is the electric near-horizon limit (3.22).

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